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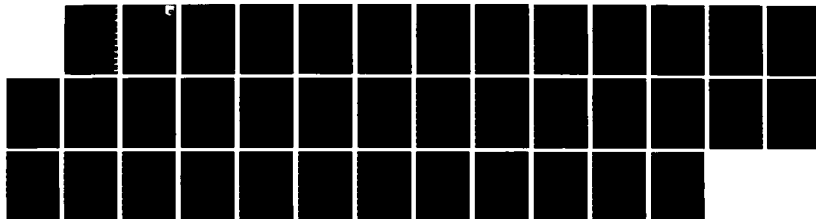
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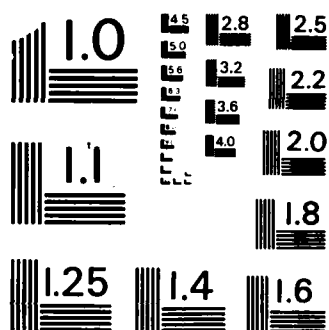
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STATIC AND DYNAMIC FORMULATION OF A
SYMMETRICALLY LAMINATED BEAM FINITE
ELEMENT FOR A MICROCOMPUTER

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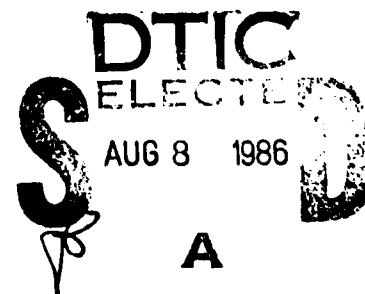
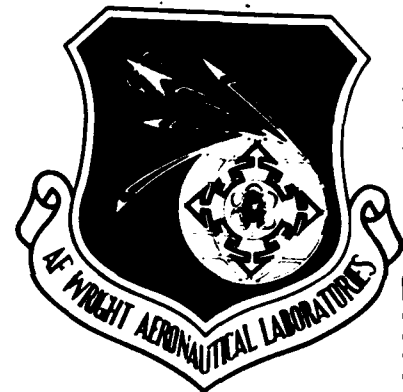
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FOREWORD

This report was prepared by T.Y. Yang and Alex T. Chen, School of Aeronautics and Astronautics, Purdue University for the Mechanics and Surface Interactions Branch (AFWAL/MLBM), Nonmetallic Materials Division, Materials Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson AFB, Ohio. The work was performed under Contract Number F33615-83-C-5076, Project Number FY1457-83-02208.

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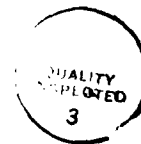


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LIST OF SYMBOLS

B_1, B_2	first and second bending modes
b	width of the beam
D_{ij}	bending stiffnesses
DS_{44}	transverse shear stiffness
d_{11}, d_{16}	compliances
E	modulus of elasticity
F	force
F_i	nodal forces (point loads and moments, etc.)
F_1, F_2	point loads at the respective beam element nodes
f_1, f_2, f_3, f_4	shape functions
Hz	Hertz
h	height, depth, or thickness of the beam or laminate
I	moment of inertia about the centroid
J	polar mass moment of inertia about the centroidal axis
k	stiffness
L	length of the beam
ℓ	length of the beam element
M_x, M_y, M_{xy}	plate bending moments
M_1, M_2	bending moments at the respective beam element nodes
m	mass per unit length of the beam element
P	applied point load
Q_x	shear force
q	nodal degrees of freedom
s	symmetric
T	kinetic energy
T_1	first torsional mode
T_1, T_2	torques at the respective beam element nodes 1 and 2

LIST OF SYMBOLS (Concluded)

T_1, T_2	rate of torque with respect to beam axis at the respective beam element nodes 1 and 2
t	time
t_i	ply thickness at the i th layer
U	potential energy
W	transverse deflection
W_b	beam transverse deflection due to bending deformation
W_{b1}, W_{b2}	transverse deflection due to bending deformation at the respective beam element nodes 1 and 2
W_s	beam transverse deflection due to shear deformation
W_{s1}, W_{s2}	transverse deflection due to shear deformation at the respective beam element nodes 1 and 2
x	x-coordinate (along the beam axis)
y	y-coordinate (along the beam width)
z	z-coordinate (along the beam thickness or depth)
α	shear coefficient
γ_x	shear angle
$\epsilon_x, \epsilon_y, \epsilon_{xy}, \epsilon_{xz}$	inplane and transverse strains
θ	ply angle
θ_{b1}, θ_{b2}	bending slope due to bending deformation at the respective beam element nodes 1 and 2
θ_{s1}, θ_{s2}	bending slope due to shear deformation at the respective beam element nodes 1 and 2
π	pi
ρ	mass per unit volume
$\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}$	inplane and transverse stresses
τ_1, τ_2	twisting angle at the respective beam element nodes 1 and 2
$\dot{\tau}_1, \dot{\tau}_2$	rate of twist with respect to beam axis at the respective beam element nodes 1 and 2
ϕ	twisting angle
ψ	shear angle
ω	harmonic frequency

SECTION I

INTRODUCTION

During the last three decades, the research and development in finite element method have grown from infancy to maturity and this method has revolutionized the methods of structural analysis and design. Such development has been well documented in the texts by, for example, Martin [1], Zienkiewicz [2], and Gallagher [3].

During the last decade, the development of calculators and desk-top mini-computers has grown at an explosive rate and it becomes an obvious trend that more simple daily structural designs will be performed using microcomputers. To cope with such trend in the rapidly growing use of microcomputers, works have currently been underway to convert some general purpose finite element programs suitable for microcomputers.

During the last two decades, the research and development of laminated composite structures has grown at an extremely rapid pace and it becomes an obvious trend that more and more composite material will be used in the design of structures when the weight and strength are of primary consideration. The fundamental development in the mechanics of composite materials has been documented by, for example, Tsai [4] and Jones [5]. The basic theory of the mechanics of composite materials, particularly for laminated plates, has been widely used in finite element formulations. Thus, it is common that existing isotropic and homogeneous finite elements also have the capability of treating laminate composite materials. A discussion on the use of finite element method in the study of composite laminated plates can be found in, for example, Reference 6. An evaluation of finite-element software for stress analysis of laminated composites was given in Reference 7.

In view of the trend of growing use of microcomputers in the common engineering office, it appears that there is a definite need for research and development work to tailor and simplify the basic formulations for laminated composite finite elements into a form suitable for programming using microcomputers. As a first step to respond to this need, in this paper, a simplified symmetrically laminated beam-type finite element is developed and programmed for a microcomputer. The element is assumed to have six degrees of freedom at each of the two ends: transverse deflection and slope due to bending and shear, respectively; and a twisting angle and its derivative with respect to beam axis.

This program implemented in a microcomputer is capable of performing stress analysis of symmetrically laminated beam structures with a single or combined effect of bending moment, twisting moment, and shear deformation, and with arbitrary loading and boundary conditions. It is also capable of performing free vibration analysis without shear deformation. For static analysis, the program has the capability of providing both numerical data and graphical plots of the distributions of displacements, bending and twisting moments, ply stresses, and the portions contributed by shear deformation. For free vibration analysis, the program gives the natural frequencies and mode shapes.

The simple homogeneous and isotropic beam finite element formulation is essentially the same as that traditionally formulated by the slope-deflection equations for beams. Such an element was extended by, among others, McCalley [8], Archer [9], and Kapur [10], to include the effect of shear deformation. Archer's element has two degrees of freedom at each of the two nodes: total transverse deflection due to combined effect of bending and shearing deformations and its derivative with respect to beam axis.

On the other hand, Kapur developed a beam element where the bending and shearing deformations were considered separately with separate associated degrees of freedom. Thus the element has four degrees of freedom at each of the two nodes: transverse deflection and its derivative with respect to the beam axis due to bending and shearing deformations, respectively. For the present laminated composite beam finite element, Kapur's type of approach was used to account for the effect of shear deformation. The homogeneous anisotropic beam theory which considered the coupling between bending and torsion was given by Lekhnitskii [11]. Such coupling was incorporated in the present formulation. A beam finite element formulation for laminated composite material with a single fiber orientation and shear deformation was given by Teh and Huang [12] for free vibration analysis. In that element, six degrees of freedom were assumed at each node: total deflection; total slope; twist derivative of bending slope with respect to beam axis; second derivative of bending slope with respect to beam axis; angular displacement and angle of twist. This paper differs from Reference 9 in that different types of degrees of freedom are assumed aiming at performing static and free vibration analyses using a microcomputer in a simpler, more efficient and general fashion.

To evaluate the present formulation, solution procedure, and program developed in this study, a series of examples, all with existing solutions for comparison, were performed using a desk-top microcomputer. For static analysis, these evaluations include: (1) an example of an isotropic, homogeneous beam with the effect of shear deformation; (2) an example of an orthotropic cross-ply laminated beam including the effect of shear deformation; (3) an example of an anisotropic laminated beam with no effect of shear deformation; and (4) an example of a quasi-isotropic laminated

beam. For free vibration analysis, the evaluations include: (1) an example of an isotropic homogeneous cantilever plate in free vibration without shear deformation; and (2) examples of anisotropic plates with varying stacking sequences in free vibration without shear deformation.

SECTION II

SYMMETRICALLY LAMINATED BEAM FINITE ELEMENT FORMULATION

In this formulation, a symmetrically laminated composite beam was considered. The beam is made of layers of orthotropic material in which the orthotropic axes of each layer may be oriented at an arbitrary angle with respect to the beam axis. In Figure 1, the positive ply orientation angle and the various parameters are defined.

2.1 Description of Element

The present symmetrically laminated beam element is described in Figure 2. The element possesses 6 degrees of freedom at each of the two nodes: the deflections due to bending W_b , the deflection due to shear deformation W_s , and their respective derivatives with respect to the x-axis $\theta_b = (-dW_b/dx)$ and $\theta_s = (-dW_s/dx)$, and the twisting angle ϕ and its derivative $d\phi/dx$. The displacement functions for W_b , W_s , and ϕ are assumed as,

$$\begin{aligned} W_b(x) &= f_1(x)W_{b1} + f_2(x)\theta_{b1} + f_3(x)W_{b2} + f_4(x)\theta_{b2} \\ W_s(x) &= f_1(x)W_{s1} + f_2(x)\theta_{s1} + f_3(x)W_{s2} + f_4(x)\theta_{s2} \\ \phi(x) &= f_1(x)\tau_1 + f_2(x)\tau_1' + f_3(x)\tau_2 + f_4(x)\tau_2' \end{aligned} \quad (1a)$$

where the shape functions are in terms of

$$\begin{aligned} f_1(x) &= 1 + 2(x/\ell)^3 - 3(x/\ell)^2, \\ f_2(x) &= -x + 2x^2/\ell - x^3/\ell^2, \\ f_3(x) &= 3(x/\ell)^2 - 2(x/\ell)^3, \\ f_4(x) &= -x^3/\ell^2 + x^2/\ell. \end{aligned} \quad (1b)$$

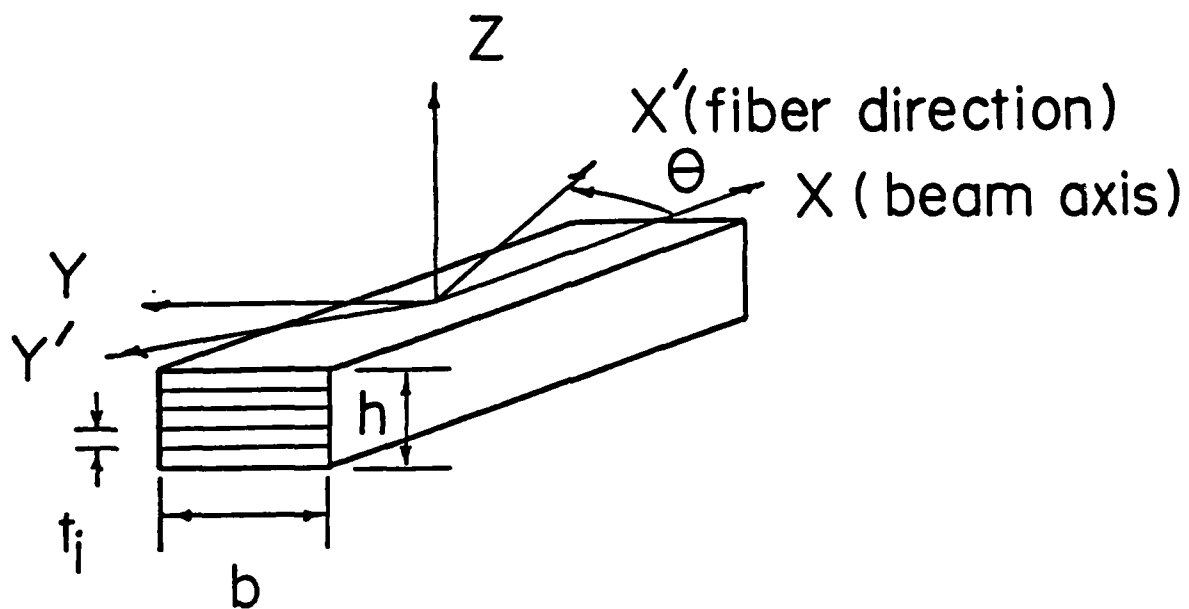


Figure 1. Positive ply orientation.

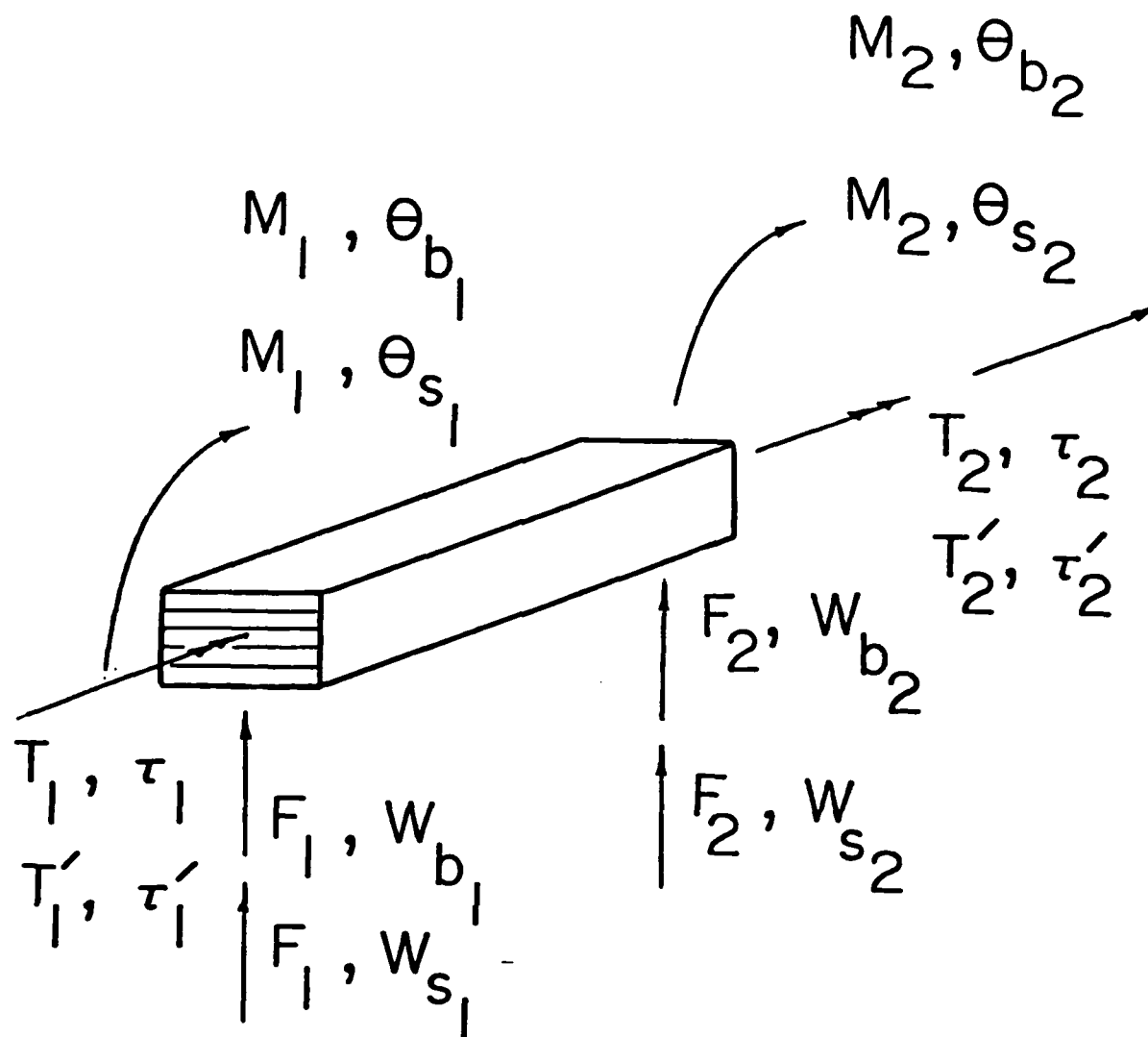


Figure 2. The 12 d.o.f. element with the effect of shear deformation.

2.2 Formulation

The stress-strain relation due to bending deformation only can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} \quad (2a)$$

where the σ 's are the stress components, ϵ 's are the strain components, and the Q_{ij} 's are the inplane ply stiffnesses.

The stress-strain relation due to transverse shear deformation is given by

$$\sigma_{xz} = Q_{44} \epsilon_{xz} \quad (2b)$$

The moment-curvature relations for an anisotropic plate due to bending deformation are given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (3a)$$

where M_x , M_y , and M_{xy} are the bending and twisting moments; k_x , k_y , and k_{xy} are the curvatures due to bending deformations alone; and D_{ij} 's are the flexural moduli. For the case of the beam, no bending moment M_y exists.

It is noted, however, that the curvature k_y is assumed to be nonzero.

Thus, the moment curvature relation may be rewritten as

$$\begin{Bmatrix} M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} - \frac{D_{12}^2}{D_{22}} & D_{16} - \frac{D_{12}D_{26}}{D_{22}} \\ D_{16} - \frac{D_{16}D_{26}}{D_{22}} & D_{66} - \frac{D_{26}^2}{D_{22}} \end{bmatrix} \begin{Bmatrix} k_x \\ k_{xy} \end{Bmatrix} \quad (3b)$$

The transverse shear force-strain relation is given by

$$Q_x = \alpha DS_{44} \gamma_x \quad (4)$$

where Q_x is the transverse shear force, α is the shear coefficient, γ_x is the transverse shear strain, and DS_{44} is the transverse shear stiffness.

The strain energy expression is given as

$$U = \frac{b}{2} \int \{M_x k_x + M_{xy} k_{xy}\} d\text{Area} + \frac{b}{2} \int \{Q_x \gamma_x\} d\text{Area} \quad (5)$$

where

$$k_x = \partial^2 W_b / \partial x^2, \quad k_{xy} = 2\partial\phi / \partial x, \quad -\gamma_x = \partial W_s / \partial x, \quad \text{and}$$

b = width of the beam.

The kinetic energy expression for a plate can be written in the general terms as

$$T = \frac{\rho}{2} \iiint \dot{w}^2(x,y,t) dx dy dz \quad (6)$$

where ρ is the mass density per unit volume. The dot represents derivative with respect to time.

For the case of the beam and in the absence of shear deformation, the deflection function may be written as, from Equations (1),

$$W(x,y) = W_b(x) + y \phi(x) \quad (7)$$

Substituting Equation (7) into Equation (6) and performing integration result in,

$$T = \frac{\rho A}{2} \int_0^L \dot{w}_b^2(x) dx + \frac{J}{2} \int_0^L \dot{\phi}^2(x) dx \quad (8)$$

where A = cross-sectional area of the beam, and J = polar mass moment of inertia about the centroidal axis.

Substituting the kinetic energy expression Equation (8) and the strain energy expression Equation (5) into the Lagrange's equation

$$F_i = \frac{d}{dt} \frac{dT}{dq_i} + \frac{dU}{dq_i} \quad (9a)$$

yields

$$\{F\} = [k]\{q\} + [m]\{\ddot{q}\} \quad (9b)$$

where $[k]$ is the stiffness matrix; $[m]$ is the consistent mass matrix, (which can also be derived in other form such as "lumped"); and $\{q\}$ is the vector for the six nodal degrees of freedom.

Assuming free vibration with simple harmonic frequency ω , Equation (9b) becomes

$$\{[k] - \omega^2[m]\} \{q\} = 0 \quad (10)$$

The stiffness matrix and the consistent mass matrix are given as follows.

MASS MATRIX

$$\begin{bmatrix}
 M_{11} & M_{12} & & & M_{15} & M_{16} & & \\
 & M_{22} & & & M_{25} & M_{26} & & \\
 & & M_{33} & M_{34} & & & M_{37} & M_{38} \\
 & & & M_{44} & & & M_{47} & M_{48} \\
 & & & & M_{55} & M_{56} & & \\
 & & & & & M_{66} & & \\
 \text{SYMMETRIC} & & & & & & M_{77} & M_{78} \\
 & & & & & & & M_{88}
 \end{bmatrix}$$

WHERE

$M_{11} = 156 F$; $M_{12} = -22L F$; $M_{15} = 54 F$; $M_{16} = 13L F$; $M_{22} = 4L^2 F$; $M_{25} = -13L F$;
 $M_{26} = -3L^2 F$; $M_{33} = 156 G$; $M_{34} = -22L G$; $M_{37} = 54 G$; $M_{38} = 13L G$; $M_{55} = 156 F$;
 $M_{56} = 22L D$; $M_{66} = 4L^2 F$; $M_{77} = 156 G$; $M_{78} = 22L G$; $M_{88} = 4L^2 G$; $G = JL/420$;
 $F = mL/420$.

J = mass polar moment of inertia per unit length of the element;

m = mass per unit length of the element.

STIFFNESS MATRIX INCLUDING SHEAR DEFORMATION

$$\begin{bmatrix}
 K11 & K12 & & & K16 & K17 & K18 & & & & & & K1C \\
 & K22 & & & K25 & K26 & K27 & K28 & & & K2B & & K2C \\
 & & K33 & K34 & & & & & K39 & K3A & & & \\
 & & & K44 & & & & & K49 & K4A & & & \\
 & & & & K55 & K56 & & K58 & & & K5B & K5C \\
 & & & & & K66 & K67 & K68 & & & K6B & K6C \\
 & & & & & & K77 & K78 & & & & K7C \\
 & & & & & & & K88 & & & K8B & K8C \\
 & & & & & & & & K99 & K9A & & \\
 & & & & & & & & & KAA & & \\
 & & & & & & & & & & KBB & KBC \\
 & & & & & & & & & & & KCC
 \end{bmatrix}$$

SYMMETRIC

where

$$\begin{aligned}
 K11 &= -K17 = K77 = 12 D11^*/L^3; \quad K12 = K18 = -K27 = -K78 = -6 D11^*/L^2; \\
 K22 &= K88 = 4 D11^*/L; \quad K28 = 2D11^*/L; \quad K16 = -K1C = -K25 = K2B = K58 = -K67 = \\
 K7C &= -K8B = 6 D66^*/5L; \quad K56 = K5C = -K6B = -KBC = -D66^*/10; \quad K66 = KCC = \\
 2 D66^* L/15; \quad K6C &= D66^* L/30; \quad K33 = -K39 = K99 = 6 S/5L; \quad K34 = K3A = -K9A = \\
 -K49 &= -A/10; \quad K44 = KAA = 2L S/15; \quad K4A = -L S/20; \quad S = \alpha DS44; \quad D11^* = b \\
 (D11-D12^2/D22); \quad D16^* &= b(D16-D12 D26/D22); \quad D66^* = b(D66-D26^2/D22)
 \end{aligned}$$

SECTION III

MICROCOMPUTER PROGRAM

The formulation of the present 12 d.o.f. symmetrically laminated composite beam finite element has been coded into a microcomputer in Basic language. For static analysis, the program computes the transverse deflections and its slope distributions due to bending and shearing deformation along the beam axis, respectively, and twist distributions due to bending deformation along the beam axis. It then computes the moment, shear force, and torque distributions along the beam axis. It finally computes the inplane normal and shearing stresses, and transverse shearing stress distributions through the laminate thickness. For free vibration analysis, the program computes the natural frequencies and the corresponding mode shapes along the beam axis.

For the static case, the program uses a symmetrically banded matrix solver which reduces both computing time and memory storage. The program also has a graphics routine which plots the distributions of the various aforementioned quantities on the microcomputer monitor screen. There are several devices which can be used to accelerate the computational speed. The first method is to compile the program using the available compilers on the market. The second method is to use a machine-dependent arithmetic chip which accelerates the computing of numbers, functions, etc. The program also allows the user to store the results in a file, which can be printed later. Similarly, the plots may be stored in a file and recalled, and a hard copy of the plot can be obtained using available "screen dumps."

For the free vibration, the program uses a Jacobi eigenproblem solver. It computes the natural frequencies and plots the mode shapes.

SECTION IV

EVALUATIVE ANALYSIS

The formulation, solution procedure, and program have been evaluated by performing the following examples with the alternative solutions for comparison.

4.1 Static Analysis

4.1.1 Homogeneous, Isotropic Cantilever Beam Under End Load P

To test the portion of the formulation which accounts for the effect of shear deformation, an example of a homogeneous, isotropic cantilever beam under end load P was first analyzed. The results obtained using one 12 d.o.f. element for the end deflection due to the effect of shear deformation only are given in Table 1 for two different values of shear coefficient α , as defined in Reference 13. For $\alpha = 0.667$, an alternative exact solution available in the text by Timoshenko [14] and a one-element solution by Archer [9] are shown in the Table. The four values for the end deflection by the four different methods are seen to be in excellent agreement. For $\alpha = 0.867$, a one-element by Archer [9] and a solution based on the energy method by Popov [15] are also shown in the table; the present solution is 4% lower than that by Popov and 17% lower than that by Archer. It is noted that the difference between the present solution and that by Archer may be due to the difference in the boundary conditions at the fixed end. In the former, it is assumed that $dW_s/dx \neq 0$, and in the latter, it is assumed that $dW/dx + \psi = 0$, where W_s is the deflection due to shear deformation; W is the total deflection; and ψ is the shear angle.

Table 1 Maximum deflection W_s in $Plh^2/4EI$ due to shear deformation only for a homogeneous isotropic cantilever beam under end load.

	Shear Coefficient α (defined in Ref. 13)	
	$\alpha=0.667$	$\alpha=0.867$
Exact Solution (14)	1.33	---
One 12 D.O.F. Element	1.33	1.0
Archer (9)	1.34	1.17
Based on Strain Energy (15)	---	1.04

4.1.2 Simply-Supported Rectangular 3 Layer (0/90/0) Cross-ply Laminated Plate with Infinite Aspect Ratio (Wide Beam) with Shear Deformation

The second example chosen was a simply supported rectangular 3 layer (0/90/0) cross ply laminated plate with infinite aspect ratio (wide beam) subjected to a sinusoidally distributed load,

$$p = p_0 \sin \frac{\pi x}{\ell} \quad (11)$$

where ℓ is the length and x is the coordinate along the span. Due to symmetry, only half of the beam need be modeled; and 1, 2, 4, 6, and 8 elements were used, respectively. The work equivalent loads based on the integration of the products of the shape functions and the distributed load were used. The center deflection due to the effect of shear deformation only was obtained and given in Table 2.

An alternative analytical solution provided by Pagano and Whitney [16] is also shown in Table 2. It is seen that the present solution practically converged to the correct answer at the 6 element level.

Table 2 Non-dimensionalized maximum deflection due to shear deformation only of a simply-supported rectangular 3 layer (0/90/0) cross-ply cross-ply laminated plate with infinite aspect ratio (wide beam) under a sinusoidal load $P = P_0 \sin \frac{\pi x}{\ell}$

Number of Elements for half of beam	Non-dimensionalized maximum deflection $\frac{w_s \alpha DS_{44}}{P_0 \ell^2}$	Pagano and Whitney (16) $\frac{w_s \alpha DS_{44}}{P_0 \ell^2}$
1	0.3599	0.4053
2	0.4463	
4	0.4014	
6	0.4057	
8	0.4056	

4.1.3 Anisotropic 16 Layer $45_4/-45_4$ Laminated Cantilever Beam Under End Load P

The third example chosen was an anisotropic 16 layer laminated cantilever beam under an end load P without the effect of shear deformation. The results for the distributions of twisting angle and deflection due to bending, respectively, along the beam length were obtained using one element only and were plotted in Figure 3. With the use of the displacement functions in Equations (1), the displacement can be interpolated anywhere within

NON-DIMENSIONAL BEAM DISPLACEMENT

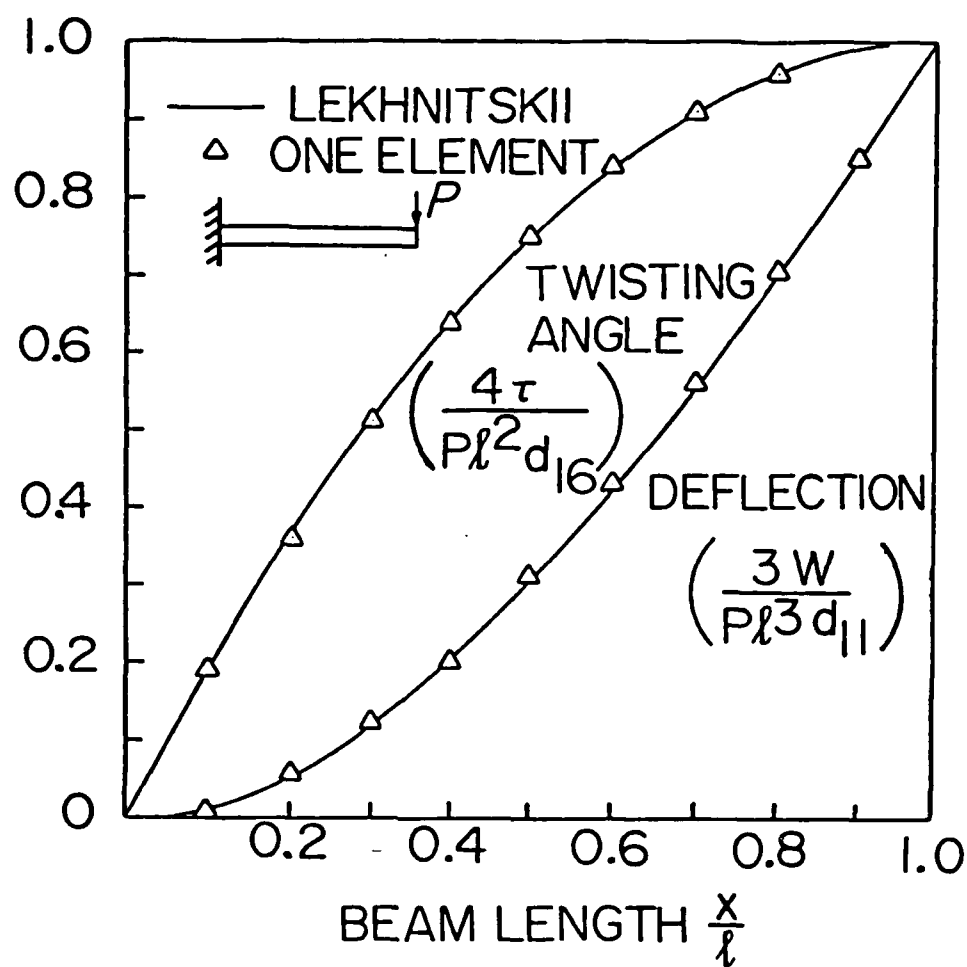


Figure 3. Distribution of deflection and twisting angle' due to bending for a 16 ply $(45_4/-45_4)_s$ anisotropic laminated cantilever beam under end load P.

the element. The twisting angle and the deflection were nondimensionalized as $(4\tau/P\ell^2 d_{16})$ and $(3W/P\ell^3 d_{11})$, respectively, where τ = twisting angle; ℓ = length of the beam, d_{11} , d_{16} are the flexural compliances; and W = deflection.

An alternative analytical solution for this problem treating the beam as homogeneous and anisotropic by Lekhnitskii [11] is also plotted in Figure 3 for comparison. The agreement is good.

The normalized inplane ply stress (σ_{xx} , σ_{xy} , and σ_{yy}) distributions through the laminate thickness were plotted in Figure 4. It is noted that force equilibrium is satisfied, and the bending moment due to end load P is recovered from summing the moments due to σ_{xx} . It is also noted that the summation of the moments due to σ_{xy} and σ_{yy} , respectively, give zero resultant moment.

4.1.4 Quasi-Isotropic $(0/90/\pm 45)_S$ Laminated Beam Under Four Point Bending

The fourth example chosen was a quasi-isotropic $(0/90/\pm 45)_S$ laminated beam under four point bending. The results for the distribution of the transverse shear stress through the laminate thickness were obtained using two elements, and were plotted in Figure 5. The shear stress has been non-dimensionalized as $(\sigma_{xz} I/Q_x h^2)$. The resultant shear force calculated from under the shear stress curve is 1.03, 3% higher than unity. The result given by Whitney [17] is also shown in the figure.

4.1.5 Anisotropic 16 Layer $(45_4/-45_4)_S$ Laminated Cantilever Beam Under End Load P with Shear Deformation

The final example chosen was the same as the previous ones but with the effect of shear deformation. The results for the distributions of

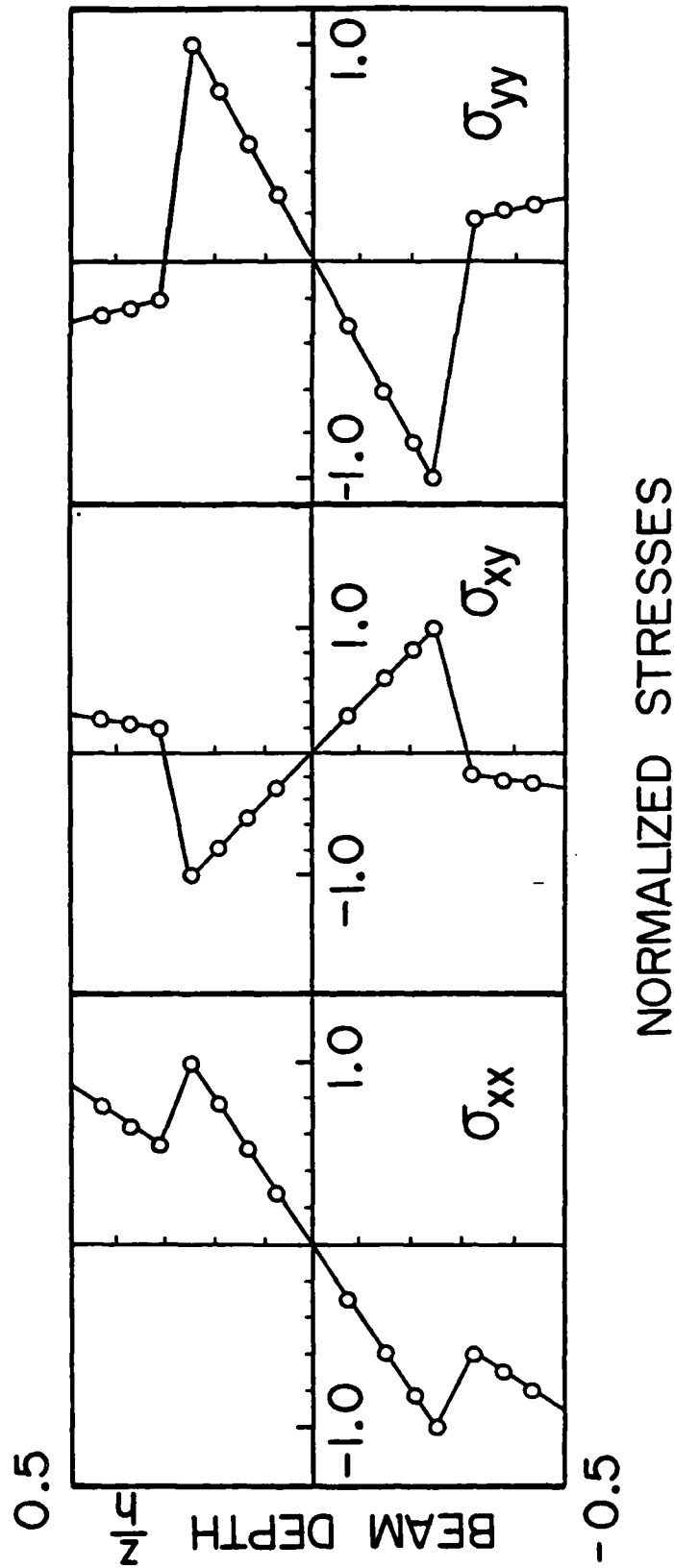


Figure 4. Distribution of inplane ply stresses through the thickness for a 16-ply $(45_q/-45_q)_s$ anisotropic laminated cantilever beam under end load P.

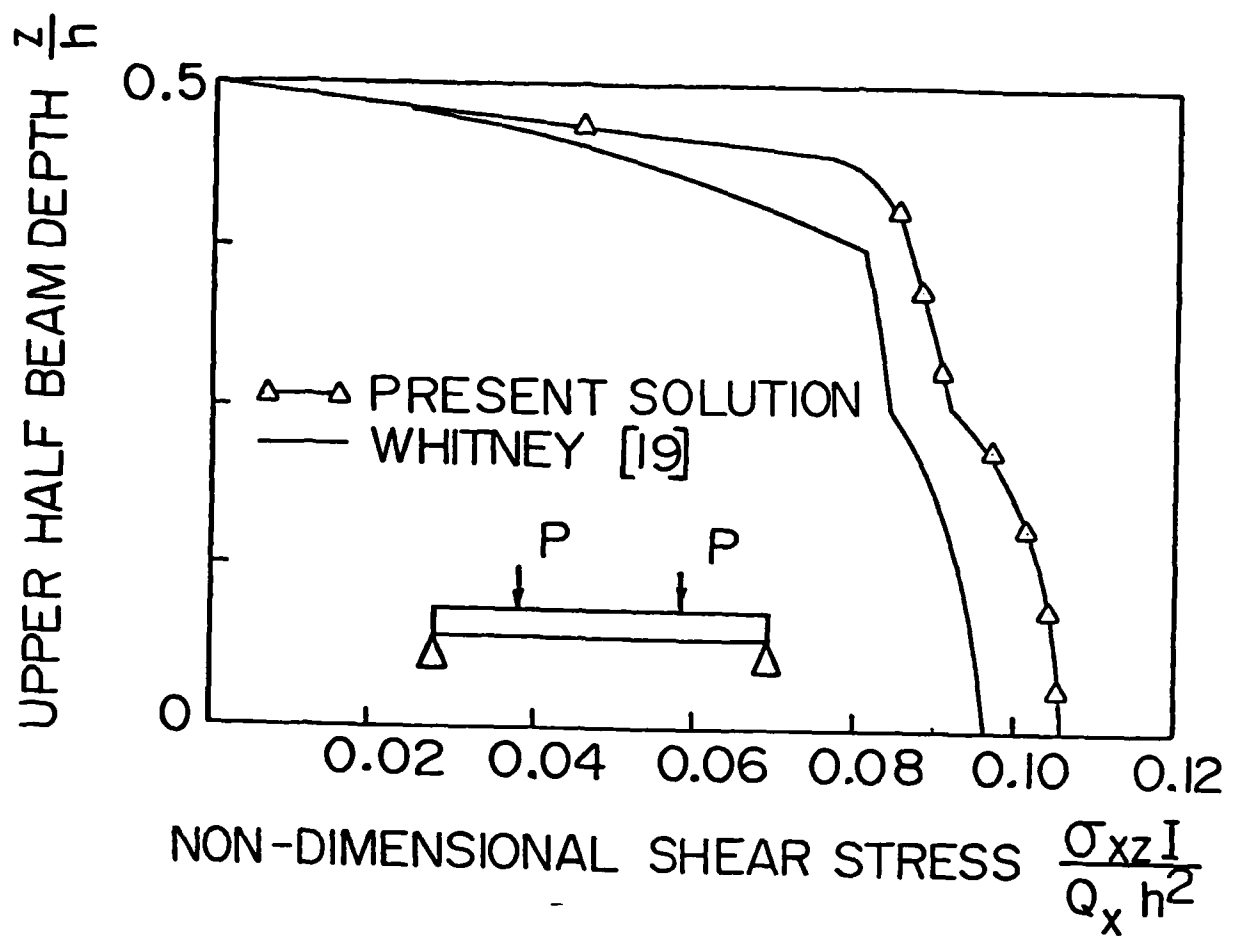


Figure 5. Distribution of transverse shear stress through the thickness of AS/4617 quasi-isotropic $(0/90/\pm 45)_s$ laminated beam under four point bending.

twisting angle and deflection due to bending as well as shear deformation along the length of the beam were obtained using one element and were plotted in Figure 6. Again, the twisting angle and the deflections were nondimensionalized the same way as were those done in Figure 3. The ply stresses are the same as those in Figure 4.

4.2 Free Vibration Analysis

4.2.1 Homogeneous, Isotropic Thin Cantilever Plate Without Shear Deformation

To test the portion of the formulation which accounts for the special case of the homogeneous, isotropic materials, an example of an aluminum cantilever plate without shear deformation was first analyzed. The results obtained using four present elements are given in Table 3. The experimental results used for comparison are given by Crawley [18]. The present solution shows good agreement for the bending frequencies. The discrepancy in torsional frequencies is due to the present modeling of the two-dimensional plate as a one-dimensional beam.

Table 3. Natural frequencies of anisotropic homogeneous aluminum cantilever plate (wide beam) without shear deformation.

Mode	Natural Frequency (Hz)	
	Four Elements	Experimental Results [18]
First Bending	38.26	37.6
First Torsion	139.47	158.3
Second Bending	240.43	234.9
Second Torsion	418.42	518.8
Third Bending	673.21	658.1

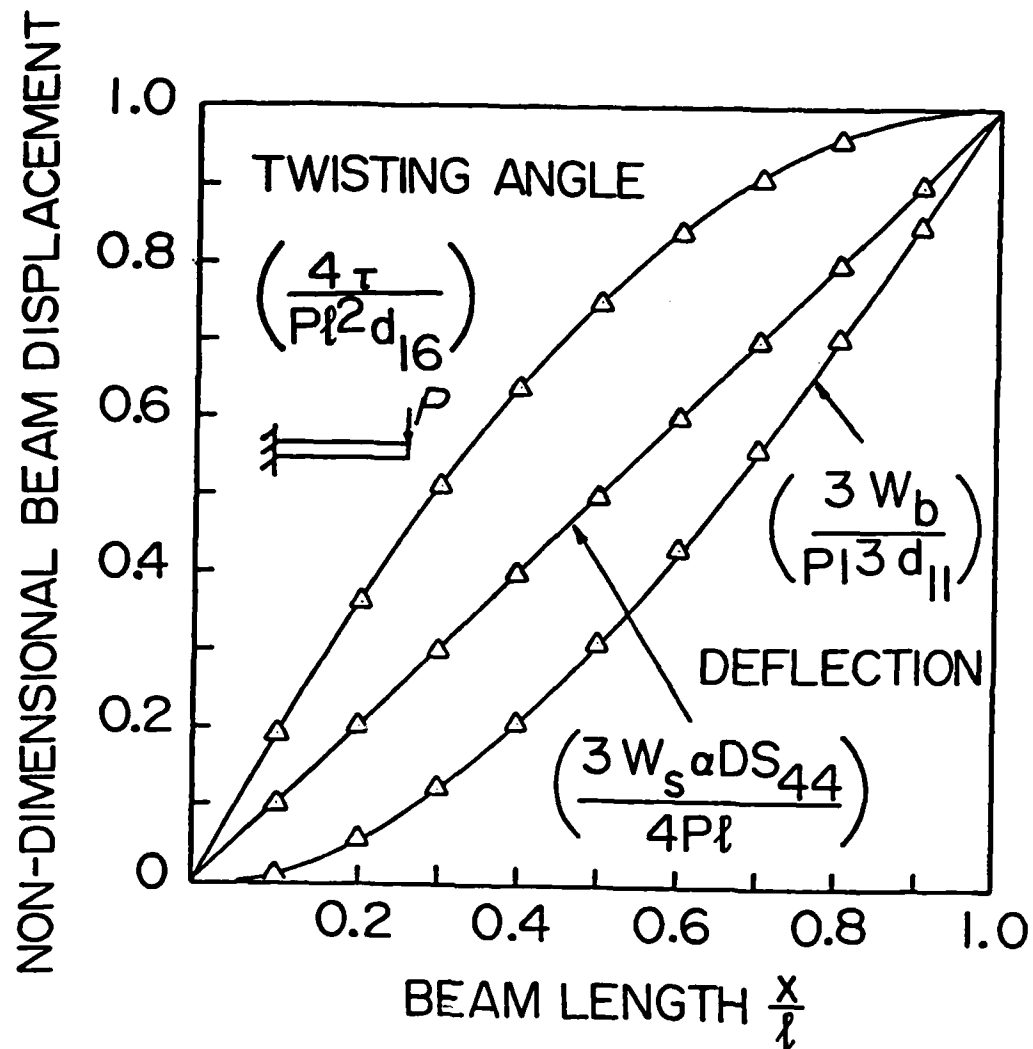


Figure 6. Distribution of deflections due to bending and shear deformation and twisting angle for a 16 ply $(45_4/-45_4)_s$ anisotropic laminated cantilever beam under end load P .

4.2.2 Thin Anisotropic Laminated Cantilever Plates Without Shear Deformation

Anisotropic laminated plates with $[\theta_2/0]_5$ and $[90/-45/45/0]_5$ stacking sequences were analyzed. The results obtained for $[\theta_2/0]_5$ stacking sequence laminated plates using a variety of numbers of the present elements are given in Table 4, for both consistent and lumped mass formulations. In Table 5, the results obtained for a $[90/-45/45/0]_5$ stacking sequence laminated plate using four elements are given, for consistent mass formulation. The experimental results used for comparison are given by Crawley [18,19]. The present solution is seen to reasonably converge at the four element level and are well converged at the eight-element level. The present solution is in good agreement with the experimental results for the bending frequencies. The percentage of discrepancies between the presently obtained torsional frequencies and the experimental values are within the approximate range of 10~20%. Again, this discrepancy may be due to the present modeling of the two-dimensional plate as a one-dimensional beam.

Table 4. Natural Frequencies of a Thin Anisotropic 6 Layer $[\theta_2 / 0]_s$ Laminated Cantilever Plate (Wide Beam) Without Shear Deformation

Lay-up Sequence	Number of Elements											Exp Results [19]
	Lumped Mass						Consistent Mass					
	1	2	4	8	10	16	1	2	4	8	10	
[0 / 90] 2 s	7.7	9.9 51.3 31.7	10.7 63.1 33.2	10.9 67.4 33.2	11.0 68.0 33.2	10.9 68.7 33.2	11.1 102.6 33.2	11.0 68.4 33.2	11.0 69.2 33.2	11.0 69.1 33.2	11.0 69.1 33.2	11.2 B1 70.5 B2 42.4 T1
[15 / 0] 2 s	5.7	7.3 35.1 41.0	7.9 49.2 40.2	8.0 51.6 40.8	8.0 51.9 40.9	8.1 52.3 41.0	8.1 84.7 43.3	8.1 52.8 41.1	8.1 52.6 41.0	8.1 52.5 41.1	8.1 52.5 41.0	9.4 B1 66.2 B2 45.8 T1
[30 / 0] 2 s	3.9	5.0 25.9 45.7	5.5 31.2 51.0	5.6 33.1 51.3	5.6 33.4 51.3	5.6 33.7 51.4	5.7 65.1 48.5	5.6 34.1 51.5	5.6 33.9 51.4	5.6 33.9 51.4	5.6 33.9 51.4	6.6 B1 40.0 B2 59.1 T1
[45 / 0] 2 s	3.2	4.1 21.0 42.1	4.4 25.7 44.9	4.5 27.4 44.9	4.5 27.7 44.9	4.5 28.0 45.0	4.6 49.6 43.0	4.6 28.4 45.0	4.5 28.2 45.0	4.5 28.1 45.0	4.5 28.1 45.0	4.8 B1 29.8 B2 51.3 T1
[60 / 0] 2 s	2.8	3.3 24.3 36.4	4.0 23.2 38.4	4.0 24.8 38.4	4.0 25.0 38.4	4.0 25.2 38.4	4.1 41.5 37.8	4.1 25.6 38.4	4.1 25.4 38.4	4.1 25.4 38.4	4.1 25.4 38.4	4.3 B1 27.1 B2 47.7 T1
[75 / 0] 2 s	2.7	3.4 17.5 32.9	3.4 22.0 34.5	3.8 23.6 34.5	3.8 23.8 34.5	3.8 24.0 34.5	3.9 34.4 38.4	3.9 24.4 34.5	3.9 24.2 34.5	3.9 24.2 34.5	3.9 24.2 34.5	3.8 B1 25.1 B2 38.9 T1
[90 / 0] 2 s	2.6	3.3 20.3 31.7	3.7 21.7 33.2	3.8 23.2 33.2	3.8 23.4 33.2	3.8 23.6 33.2	3.8 37.6 33.2	3.8 24.0 33.2	3.8 23.8 33.2	3.8 23.8 33.2	3.8 23.8 33.2	3.7 B1 24.3 B2 38.2 T1

B1, B2, and T1 mean respectively first bending, second bending, and first torsional modes.

Table 5. Natural Frequencies of an Anisotropic 8-Layer $[90/-45/45/0]_s$
Cantilever Plate

Mode	Natural Frequencies (Hz)	
	Four Elements	Experimental Results [18]
First Bending	52.407	48.6
First Torsion	138.92	169.0
Second Bending	327.73	303.0
Second Torsion	417.59	554.0

SECTION V

CONCLUDING REMARKS

In this paper, a 12 d.o.f. symmetrically laminated beam finite element, the associated solution procedure, and a computer program have been developed for the stress and vibration analyses using microcomputers. The simplicity and efficiency of this development have been evaluated through a series of examples. The effect of shear deformation for a homogeneous and isotropic beam, the effect of shear deformation for an orthotropic (0/90/0) laminated beam, the effect of bending deformation for an anisotropic laminated beam have all been verified through comparison of results with alternative existing solutions. The distribution of transverse shearing stress through the thickness for a quasi-isotropic laminated beam has also been compared. The natural frequencies of an isotropic homogeneous plate and those of an anisotropic symmetrically laminated cantilever plate have been compared.

The program was written in Basic language and matrices were stored in the form of half-band for the static analysis. For the free vibration analysis, the Jacobi eigenproblem solver was used. To expedite the computation, the program can be compiled using a compiler. A hard copy of the plots can be obtained using the available "screen dumps". For static analysis, the program can be used to obtain numerical data and graphical plots of the distributions of deflections, twisting angles, shear force, bending moment, twisting moment along the beam, ply stresses through the thickness, and the portions contributed by shear deformation. For free vibration, the program computes the natural frequencies, and plots the mode shapes.

The main goal of the present development is to promote the use of composite materials in an easy and efficient manner by the daily practicing engineers using desk-top microcomputers.

Further developments in microcomputers to analyze two dimensional truss and frame type structures, including the effects of buckling and dynamics, are currently underway.

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